Folding Algorithms and Kempe's Universality Theorem

C. Rose Morris-Wright

Brandeis University

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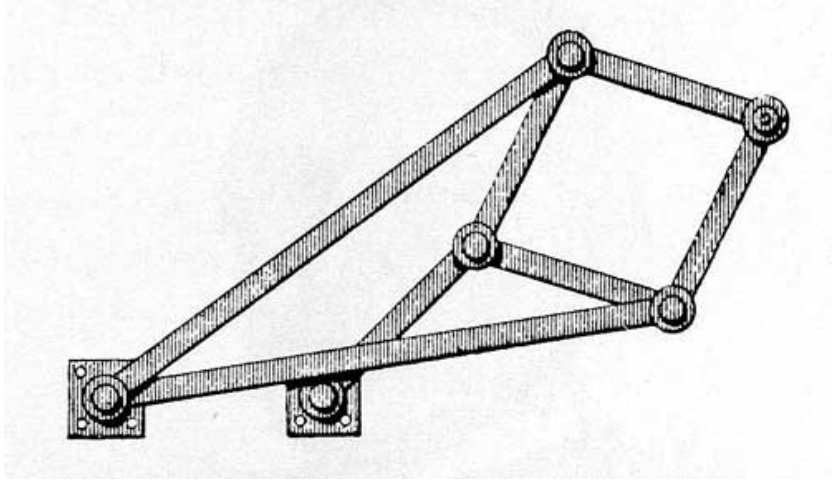
Outline

Origami and 2D folding

- History and applications

Linkages and 1D folding

- History and applications
- Some definitions
- Questions



Kempe's Universality Theorem

- Statement of theorem

The Beginning of Origami

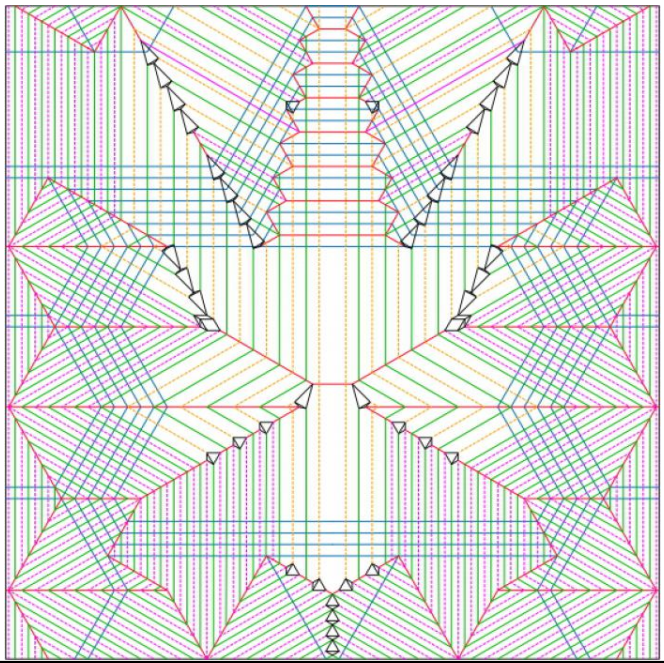
- Ancient art of paper folding, beginning in Japan around 100 A.D, just after the invention of paper.
- Many developments have been made over the past 20 years with the advent of computers.

[a]

[a] Image from langorigami.com

The Bug Wars

- Competition among origami artists to see who could make the creature with the most legs.
- Solved when Robert Lang implemented the tree algorithm which gave a crease pattern that would produce a base with the desired number of flaps.



[1]Images from langorigami.com

Further applications

1D Folding

- The study of folding in one dimension started as a way to model industrial apparatus.
- It has developed to other applications including protein folding.

Linkages

- A **Linkage** is a collection of fixed length 1D segments joined at their endpoints to form a graph
- Endpoints are called **verticies** or **joints**
- Segments are called **links**
- Some verticies are **fixed** or **pinned**

Some more definitions

- Linkages are usually considered in some ambient space, usually $\mathbb{R}^2$ or $\mathbb{R}^3$.
- A **configuration** is a specification of the location of all the verticies.
- A **configuration space** is the space of all possible configurations.

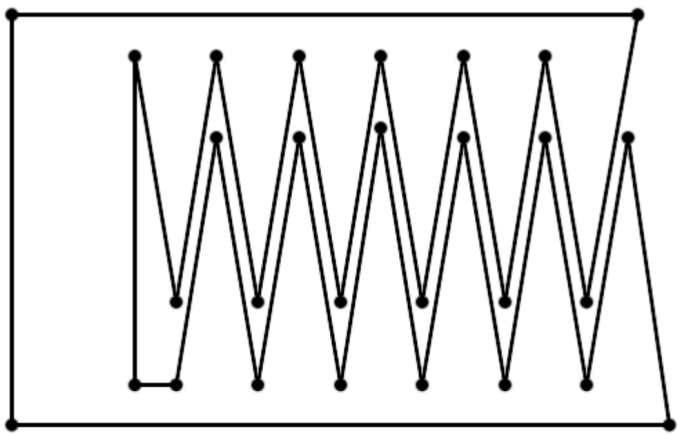
Common Questions about Linkages

Most questions that mathematicians want to answer about linkages have to do with the configuration space. For example if you might ask if the configuration space of a particular linkage is

- connected?
- a manifold?
- parameterizable?
- contains a particular configuration?

Locked Linkages

Given a non self-intersecting linkage, are the flat/convex configurations in the configuration space and connected to a particular configuration.



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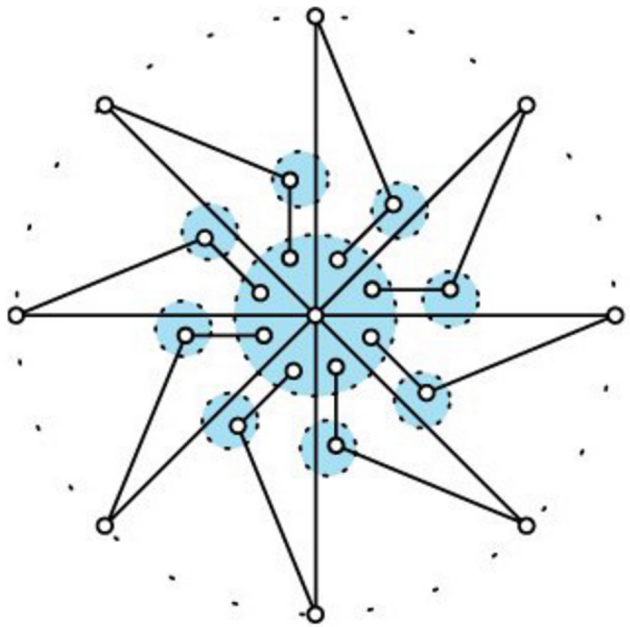

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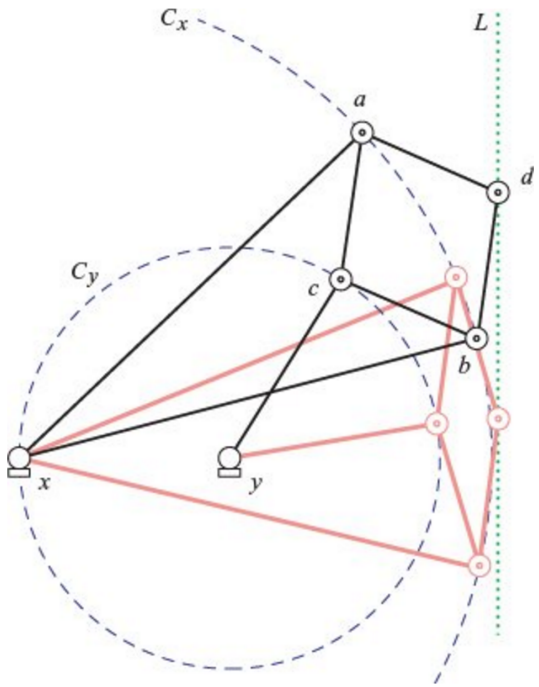
[2] Image from MIT open courseware

[3] Image from Demaine, Erik D., O'Rourke, Joseph. *Geometric folding algorithms linkages, origami, polyhedra*

1 dimensional configuration spaces

- If a linkage has a 1D configuration space then we can consider the curve traced out by one of the free verticies.
- Kempe's Theorem is about the construction of linkages that trace out a given curve.



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[4]Image from Demaine, Erik D., O'Rourke, Joseph. *Geometric folding algorithms linkages, origami, polyhedra*

Theorem

Theorem

Let C be a bounded portion of an algebraic curve in the plane(i.e. the intersection of the zero set of a polynomial $p(x, y)$ with a closed disc). Then there exists a planar linkages such that the configuration space of one vertex is precisely C.

First introduced by Alfred Kempe in 1876. Final proof with all errors corrected by Kapovich and Millson in 2002.
Famously dubbed the theorem which says there is a linkage which "signs your name".

Further reading

- Demaine, Erik D., O'Rourke, Joseph. *Geometric folding algorithms linkages, origami, polyhedra*. Cambridge University Press, New York, NY, 2007
- Demain Erik D. http://erikdemaine.org/
- Gould Vannessa, *Between the Folds*. Green Fuse Films, Brooklyn, New York 2010
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- O'Rourke Joseph, Satyan L. Devadoss. *Discrete and Computational Geometry* Princeton University Press, Princeton, New Jersey 2011